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Nonlinear development of modulated two-dimensional gravity wavetrains in deep water

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Abstract. This paper gives an analytic description of the nonlinear development of the modulational instability with the two-dimensional nonlinear (cubic) Schrödinger equation governing the evolution of a weakly nonlinear, deep-water gravity wavetrain and tries to shed some light on the previous numerical and analytical results on this problem. Toward this objective, we first develop a proper formulation of the method of multiple scales to describe the long-time behaviour of the linearly unstable modulation near the threshold for linear instability for the two-dimensional case and investigate whether the nonlinear development of this modulation indicates a quasi-periodic motion in the latter case.

1. Introduction

Poincaré's famous theorem on recurrent motions in Hamiltonian systems states that in a bounded Hamiltonian system with a finite number of degrees of freedom, almost all initial phase points will evolve in time such that they will return to an arbitrarily small neighbourhood of the initial point after a finite time interval. At first sight this theorem would appear to rule out recurrent motions for continuum systems which (at least formally) possess an infinite number of degrees of freedom. Thyagaraja [1, 2] exhibited, however, some examples of one-dimensional nonlinear wave systems which appear to have only a finite number of 'effective' degrees of freedom which interact nonlinearly and exchange energy perpetually with each other, and so exhibit recurrence of states. These results gave a new insight into the numerical investigations of Yuen and Fergusson [3] on the dynamical structure of nonlinearly saturated, spatially-periodic solutions of the one-dimensional cubic nonlinear Schrödinger equation:

$$\mathrm{i}\frac{\partial\Phi}{\partial t}+\frac{1}{2}\frac{\partial^2\Phi}{\partial x^2}+\kappa(|\Phi|^2-|\Phi_0|^2)\Phi=0 \qquad (1)$$

where Φ_0 is an arbitrary reference amplitude. Yuen and Fergusson [3] numerically evolved linearly unstable initial data. (The instability in question is the so-called side-band instability discovered by Benjamin and Feir [4].) The long-time behaviour of those solutions revealed that the energy sharing effectively occurred among a finite number of modes. Furthermore, the solution 'reconstructed' itself after appearing to show a tendency to 'break up' due to linear instability. The discovery of the failure to thermalize as well as the tendency to recur goes back to the early work of Fermi *et al* [5] on the oscillations of an anharmonic lattice, and the long-time, periodic behaviour of a nonlinear system has become known as the Fermi-Pasta-Ulam recurrence phenomenon. Lake *et al* [6] in a series of careful experiments, showed that recurrence

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is a generic phenomenon in situations adequately described by the one-dimensional nonlinear Schrödinger equation.

The essence of the recurrence phenomenon is that linearly unstable modes nonlinearly evolve into a superperiodic state on a sufficiently long timescale. When this superperiodicity is common to all Fourier modes, the initial conditions will be reproduced every once in a while. Janssen [7] and Rowlands [8] made an attempt to show analytically that the long-time behaviour of spatially periodic solutions of the one-dimensional nonlinear Schrödinger equation is periodic. This was based on the well known linear-stability result that a finite-amplitude uniform wavetrain is unstable to infinitesimal modulational perturbations with sufficiently long wavelengths while it is stable for perturbations with short wavelengths so that a threshold for instability exists. Near the threshold for instability, Janssen [7] and Rowlands [8] then obtained the long-time behaviour of the unstable modulation by means of the method of multiple scales, and tried to show that the nonlinear effects stabilize the linearly unstable modulation and produce a periodic motion. However, Janssen [7] and Rowlands [8] did not completely succeed with their objective because they did not formulate the method of multiple scales properly to treat the instability threshold region, and hence their analytic results could not be expressed in a very meaningful way. Infeld [9] gave a more general calculation in which the restriction to the vicinity of the instability threshold region was relaxed. However, Infeld [9] ignored the second and higher harmonics.

The evolution of a weakly nonlinear, deep water gravity wavetrain subjected to a two-dimensional modulation is described by the following two-dimensional nonlinear Schrödinger equation [10, 11]:

$$\mathrm{i}\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left(\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}\right)+\kappa(|\Phi|^2-|\Phi_0|^2)\Phi=0. \tag{2}$$

This equation is obtained from the more common form, in which the last term is absent, via the substitution $\phi=\mathrm{e}^{-\mathrm{i}\kappa|\Phi_0|^2 t}\Phi$. Yuen and Fergusson [12] and Martin and Yuen [13] obtained numerical solutions of this equation for spatially periodic boundary conditions and showed the existence again of the connection between the linear instability aspects and the long-time evolution of certain classes of solutions of this equation. They showed, in particular, that the long-time evolution of the linearly unstable evolution is composed of the growth and decay of all the harmonics of the initial perturbation that lie within the unstable region, each one alternately dominating the evolution. However, because of the negative transverse dispersion the instability region in the wavenumber space for this equation is unbounded [14], and consequently Martin and Yuen [13] found that the energy initially contained in low modes would leak to unstable arbitrarily higher harmonics. Thus, for long times, the energy sharing occurred among an arbitrarily high number of modes. An analytic explanation of this phenomenon was recently given by Shivamoggi and Mohapatra [15], who showed that this nonlinear system can be described as effectively possessing an arbitrarily high number of degrees of freedom so that the overall motion can at best be only quasi-recurrent.

In this paper, we formulate the method of multiple scales properly to describe the long-time behaviour of spatially periodic solutions of the two-dimensional nonlinear (cubic) Schrödinger equation (2). This involves inserting in the solutions near the threshold for linear instability, an explicit detuning parameter χ [16] i.e., $\kappa=\kappa_c(1+\Delta^2\chi+O(\Delta^3))$, $\Delta\ll 1$, corresponding to equation (7) in Janssen's [7] paper. (One could then demonstrate, showing in an elegant way Janssen's [7] original objective, for the one-dimensional case, that the nonlinear effects stabilize the linearly unstable modulation and produce a periodic motion.) Since the threshold for linear instability in the two-dimensional case exhibits two distinct branches [14] we will construct solutions separately near each branch. The method of treatment for the two-dimensional case needs to be modified, however. For one thing, we will now have to perturb the wavenumbers instead of the nonlinearity parameter κ. Besides, we will have to write different types of expansions for the two branches of the linear instability threshold. We will investigate the nonlinear effects on a linearly unstable modulation near the threshold. Since one of the two branches of the linear instability threshold for the two-dimensional case reduces to the instability threshold of the one-dimensional case in the appropriate limit, the nonlinear evolution near this branch is one-dimension like. While the other instability-threshold branch is peculiar to the two-dimensional case so that the nonlinear evolution near this branch reflects features that are peculiar to the two-dimensional case.

2. Modulational instability for the two-dimensional case

In order to investigate the modulational instability of the wavetrain whose evolution is governed by equation (2), we put

$$\phi = \rho^{1/2} \, e^{i\sigma} \tag{3}$$

so that equation (2) gives

$$\rho_t + (\rho\sigma_x)_x - (\rho\sigma_y)_y = 0 \tag{4}$$

$$\sigma_t + \frac{1}{2}(\sigma_x^2 - \sigma_y^2) + \frac{1}{8\rho^2}(\rho_x^2 - \rho_y^2) - \frac{1}{4\rho}(\rho_{xx} - \rho_{yy}) - \kappa(\rho - \rho_0) = 0. \tag{5}$$

In order to perform the linear stability analysis, we put

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} \rho_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \sigma_1 \end{pmatrix} e^{i(k_1 x + k_2 y - \omega t)} \tag{6}$$

where k_1, k_2 are the wavenumber components and ω is the frequency of the modulation. Assuming that $|\rho_1| \ll |\rho_0|$, and keeping only the terms linear in ρ_1 and σ_1, we obtain from equations (4) and (5),

$$\omega^2 = \tfrac{1}{4}(k_1^2 - k_2^2)(k_1^2 - k_2^2 - 4\kappa\rho_0). \tag{7}$$

Thus, if $\kappa > 0$, ω^2 is negative in the region of the k_1, k_2 plane bounded by the lines $k_1 = \pm k_2$ and the hyperbolas $k_1^2 - k_2^2 = 4\kappa\rho_0$ (see figure 1).

We will now consider the nonlinear development of the initially linearly unstable modulation. For this purpose, we consider the initial-value problem for modulations with wavenumbers near the threshold for instability. For the two-dimensional modulation, the latter are given by the following branches

$$\left.\begin{aligned} k_1^2 - k_2^2 &= 4\kappa\rho_0 \\ k_1 &= \pm k_2 \end{aligned}\right\}.$$

The first set of branches reduces to the instability threshold $k_1 = \pm\sqrt{4\kappa\rho_0}$ for the one-dimensional modulation, in the limit $k_2 \to 0$. The second set of branches is peculiar for the two-dimensional modulation and does not exist for the one-dimensional modulation. We will now construct solutions separately near each of the branches.

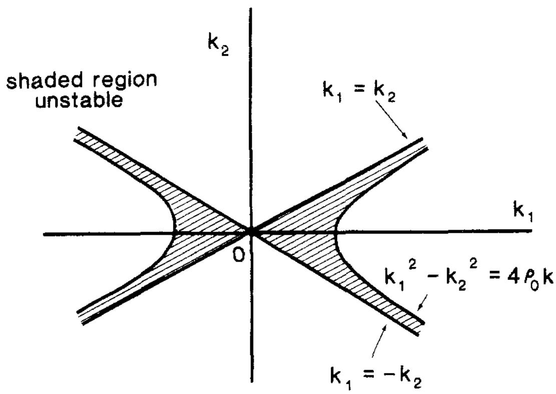

Figure 1. Modulational instability thresholds for the two-dimensional case.

3. Solution near the hyperbolic branches

We look for a solution here of the following form:

$$\begin{aligned} \rho(x, y, t) &= \rho_0 + \varepsilon\rho_1(x, y, \tau) + \varepsilon^2\rho_2(x, y, \tau) + \ldots \\ \sigma(x, y, t) &= \varepsilon\sigma_1(x, y, \tau) + \varepsilon^2\sigma_2(x, y, \tau) + \ldots \\ k_1^2 - k_2^2 &= 4\rho_0\kappa + \varepsilon^2\chi + \ldots \end{aligned} \tag{8}$$

where ε is a small parameter that characterizes the departure of $k_1^2 - k_2^2$ from the linear stability threshold value $4\rho_0\kappa$, and $\tau = \varepsilon t$ is a slow timescale characterizing slow time evolutions near the stability threshold. We have introduced an explicit detuning parameter χ.

Substituting (8) into equations (4) and (5), we obtain the following systems of equations to various orders in ε:

$$O(\varepsilon^n)\colon L\begin{pmatrix}\rho_n \\ \sigma_n\end{pmatrix} = S_n(\rho_0, \rho_1, \ldots, \rho_{n-1}; \sigma_1, \ldots, \sigma_{n-1}) \qquad n = 1, 2, \ldots \tag{9}$$

where

$$L \equiv \begin{bmatrix} 0 & \rho_0\left(\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2}\right) \\ \dfrac{1}{4}\left(\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2}\right) + \kappa\rho_0 & 0 \end{bmatrix}$$

and the function S_n depends on the solutions up to $O(\varepsilon^{n-1})$.

For $O(\varepsilon)$, we obtain

$$L\begin{pmatrix}\rho_1 \\ \sigma_1\end{pmatrix} = 0 \tag{10}$$

the solution for which corresponds to the neutrally stable case of the linear problem:

$$\rho_1 = a(\tau)\,e^{i(k_1 x + k_2 y)} + \text{CC} \qquad \sigma_1 = \alpha(\tau) \qquad k_1^2 - k_2^2 = 4\rho_0\kappa \tag{11}$$

where CC means complex conjugate.

Note that equation (10) actually allows one to take for σ_1,

$$\sigma_1 = \alpha(\tau)\,e^{i(mk_1 x + nk_2 y)} + \text{CC}$$

where $mk_1 = nk_2$. However, with the exception of the case $m = n = 0$, this infinite-dimensional null space leads only to a static case by eliminating the time dependence of $a(\tau)$. (The latter situation arises in course of removing the secular terms in the $O(\varepsilon^3)$ problem.)

Using (11), to $O(\varepsilon^2)$, we obtain

$$L\begin{pmatrix}\rho_2\\ \sigma_2\end{pmatrix} = \begin{bmatrix} -\dfrac{\mathrm{d}a}{\mathrm{d}\tau}\,\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc} \\ -\kappa|a|^2 + \rho_0\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau} + (-\frac{3}{2}\kappa a^2\,\mathrm{e}^{2\mathrm{i}(k_1x+k_2y)} + \mathrm{cc})\end{bmatrix} \tag{12}$$

from which

$$\begin{aligned}\rho_2 &= \frac{1}{\kappa}\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} - |a|^2 + \frac{a^2}{2\rho_0}\,\mathrm{e}^{2\mathrm{i}(k_1x+k_2y)} + \mathrm{cc} \\ \sigma_2 &= \frac{1}{4\rho_0^2\kappa}\frac{\mathrm{d}a}{\mathrm{d}\tau}\,\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc}.\end{aligned} \tag{13}$$

Using (11) and (13), to $O(\varepsilon^3)$, we obtain

$$L\begin{pmatrix}\rho_3\\ \sigma_3\end{pmatrix} = \begin{bmatrix} -\dfrac{1}{\kappa}\dfrac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} + \dfrac{\mathrm{d}|a|^2}{\mathrm{d}\tau} \\ \left[\dfrac{1}{4\rho_0\kappa}\dfrac{\mathrm{d}^2a}{\mathrm{d}\tau^2} + \dfrac{1}{4}\chi a + \dfrac{\kappa}{2\rho_0}|a|^2 a \right. \\ \left. -2\kappa a^2\left(\dfrac{1}{\kappa}\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau} - |a|^2\right)\right]\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc}\end{bmatrix} + \text{non-secular terms.} \tag{14}$$

Removal of the secular terms in the first member of equation (14) requires

$$\frac{1}{\kappa}\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} - |a|^2 = \text{constant.} \tag{15}$$

Let us take the constant above to be zero. Removal of the secular terms in the second member of equation (14), then, requires

$$\frac{\mathrm{d}^2a}{\mathrm{d}\tau^2} + (\rho_0\kappa\chi + 2\kappa^2|a|^2)a = 0. \tag{16}$$

If we impose the following initial conditions,

$$\tau = 0 : a = A \qquad \frac{\mathrm{d}a}{\mathrm{d}\tau} = 0 \tag{17}$$

and take a to be real, we obtain from equation (16),

$$\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 = \kappa^2(A^2 - a^2)(a^2 - \beta) \tag{18}$$

where

$$\beta = -\frac{\rho_0\chi}{\kappa} - A^2.$$

Equation (18) shows that a is bounded and oscillates between A and $\sqrt{\beta}$ if $\beta > 0$ and oscillates between 0 and A if $\beta < 0$. This demonstrates the nonlinear saturation of the linearly unstable modulation ($\chi < 0$) near the hyperbolic branches of the linear instability threshold. Observe the elegance given to the argument by the detuning parameter χ present in the above formulation.

4. Solution near the linear branches

We look for a solution here of the following form:

$$\begin{aligned} \rho(x, y, t) &= \rho_0 + \varepsilon\rho_1(x, y, \tau) + \varepsilon^2\rho_2(x, y, \tau) + \dots \\ \sigma(x, y, t) &= \varepsilon\sigma_1(x, y, \tau) + \varepsilon^2\sigma_2(x, y, \tau) + \dots \\ k_1^2 - k_2^2 &= \varepsilon\chi_1 + \varepsilon^2\chi_2 + \dots \end{aligned} \tag{19}$$

where $\tau = \varepsilon t$.

Substituting (19) into equations (4) and (5), we obtain to $O(\varepsilon)$:

$$L\begin{pmatrix}\rho_1 \\ \sigma_1\end{pmatrix} = 0 \tag{20}$$

where L is defined in equation (9). We obtain from equation (20),

$$\rho_1 = 0 \qquad \sigma_1 = \alpha(\tau)\,\mathrm{e}^{\mathrm{i}(k_1x + k_2y)} + \mathrm{cc} \qquad k_1^2 - k_2^2 = 0. \tag{21}$$

Using (21), to $O(\varepsilon^2)$, we obtain

$$L\begin{pmatrix}\rho_2 \\ \sigma_2\end{pmatrix} = \begin{bmatrix} \alpha\rho_0\chi_1\,\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc} \\ \rho_0\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau}\,\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc} \end{bmatrix}. \tag{22}$$

Removal of the secular terms in the first member of equation (22) requires

$$\chi_1 = 0. \tag{23}$$

Using (23), we obtain from equation (22),

$$\sigma_2 = 0 \qquad \rho_2 = \frac{1}{\kappa}\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\,\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc}. \tag{24}$$

Using (21), (23) and (24), we obtain, to $O(\varepsilon^3)$:

$$L\begin{pmatrix}\rho_3 \\ \sigma_3\end{pmatrix} = \begin{bmatrix} \left(\rho_0\chi_2\alpha - \dfrac{1}{\kappa}\dfrac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2}\right)\mathrm{e}^{\mathrm{i}(k_1x+k_2y)} + \mathrm{cc} \\ 0 \end{bmatrix}. \tag{25}$$

Removal of the secular terms in equation (25) requires

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} - (\kappa\rho_0\chi_2)\alpha = 0. \tag{26}$$

Equation (26) shows that the nonlinearities, to $O(\varepsilon^3)$, have no effect on the linearly unstable modulation near the linear branches of the instability threshold. It is quite likely that a nonlinear effect may show up at $O(\varepsilon^5)$, but then it would be very weak.

5. Discussion

In this paper, we have given an analytic description of the nonlinear development of the modulational instability with the two-dimensional nonlinear Schrödinger equation governing the evolution of a weakly nonlinear, deep-water gravity wavetrain. In order to accomplish this, we formulated the method of multiple scales in a proper way to describe the long-time behaviour of the linearly unstable modulation near the thresholds for linear instability. This involved writing different types of expansions for the two branches of the linear-instability threshold and introducing an explicit detuning parameter. We then investigated the nonlinear development of this two-dimensional modulation and found that it was one-dimension-like near the hyperbolic branches of the threshold while the nonlinearities had a very weak effect, if any, on the evolution of the modulation near the linear branches of the threshold. Since the latter branches are peculiar to the two-dimensional modulation, the above result showing very weak, if any, nonlinear influence on the latter is unique to the two-dimensional case and hence appears to shed some light on the nonlinear development of a linearly unstable modulation into a quasi-periodic motion as indicated by the numerical results of Yuen and Ferguson [12] and Martin and Yuen [13] and on the occurrence of quasi-recurrent motions for the two-dimensional modulated gravity wavetrains as indicated by the analytical results of Shivamoggi and Mohapatra [15].

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